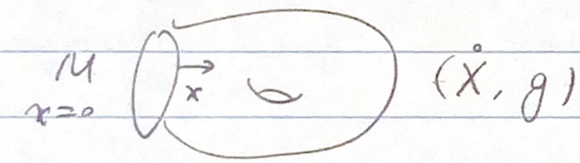
Oct 14

Fefferman – Graham expansion and Dirichlet-to-Neumann map of conformally compact Einstein metrics.

§1. Introduction

1.1 conformal geometry



X^{n+1}

- Boundary defining function : x smooth
  $x = 0$ on M
  $x > 0$ on $\mathring{X}$
  $|dx| \neq 0$

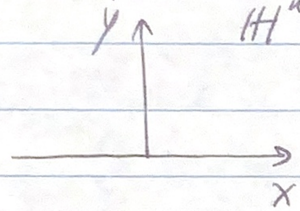
- Conformal compactification
  if $\bar{g} = x^2 g$ is a metric on $\bar{X}$, extend cont. to M

$$\bar{g}|_{TM} = x^2 g|_{TM} = h$$

defines a conformal class $[h]$
known as <u>conformal infinity</u>

g is conformally compact.

- Examples



$\mathbb{H}^{n+1}$

$$g = \frac{dx^2 + dy^2}{y^2}$$

$$y^2 g = dx^2 + dy^2 \qquad h = dx^2$$

$\mathbb{B}^{n+1}$ with $g = \frac{4\, g_{euc}}{(1-r^2)^2}$

bdf $\rho = \frac{1-r^2}{1+r^2}$

$[\bar{g}|_{TM}] = [g_{S^n}]$ ← round metric

§2. CCE. (conformally compact Einstein)

g such that $Ric_g = -ng$ on $\mathring{X}$.

$$\text{(BVP)} \quad \begin{cases} Ric_g + ng = 0 & \text{on } \mathring{X} \\ x^2 g|_{TM} \in [h] & \text{on } M \end{cases}$$

↑ prescribed conf. inf.

§2. Properties of CCE metrics.

2.1 Asymptotically hyperbolic

sec curv $-K = -1$

sec curv $-K_g \to -1$ as $x \to 0$.

- asymptotics of Rm is given by

$$Rm = -\frac{K}{2} g^2 + O(x^{-3})$$

$$Ric = -Kng + O(x^{-1})$$

Note that $g = x^{-2}\bar{g} \to g^2 = O(x^{-4})$.

In the equations above $K = |dx|^2_{\bar{g}}$ is related to the sectional curvature by a minus sign.

- $\exists\, x$ s.t. $|dx|^2_{\bar{g}} = 1$. This is called a <u>special boundary def. func</u>

2.2. Existence of CCE metrics.

[GL] <u>Thm</u> Given a metric h on S^n that is $C^{2,\alpha}$ closed to g_{S^n} (round metric), there exists a CCE metric g s.t.

$$x^2 g|_{TM} \in [h].$$

idea: The (BVP) is solvable near g_{S^n}.
$Ric_g + ng = 0$ is a Laplace type operator in the modified harmonic gauge.
(DeTurck's trick).

To be more precise, consider

$$\begin{cases} Ric_g + ng = 0 \\ \delta_g G_g(t) = 0 \end{cases}$$ (gauge fixing)

δ_g: -divergence; G_g: gravitational operator; t: auxalrary metric, $C^{2,\alpha}$ closed to g

where g is asymptotically hyperbolic:

$$g - g_{hyp} \in C^{2,\alpha}$$

This is equivalent to

[GL] $$Q(g,t) = \underbrace{Ric_g + ng}_{\text{Einstein tensor}} - \underbrace{\Phi(g,t)}_{\text{modification via gauge fixing}} = 0$$

Q: The modified Einstein tensor

Linearization w.r.t the first arguement

$$D_1 Q(g,t)$$

is a Laplace type operator. So we may compute its indicial roots and figure out when it is an isom.

This leads to solution of (BVP).
Using elliptic regularity (Laplace type op), we also get regularity on the metric. e.g. $t-g$ is C^∞ small.

2.3 CCE metrics admit Fefferman - Graham exp. in a neighbourhood of M.

- $g = \dfrac{dx^2 + h_x}{x^2}$ geodesic normal form

[FG] $h_x = \underbrace{h_0 + x^2 h_2 + \cdots + x^{n-1} h_{n-1}}_{\text{even terms}} + x^n h_n + \cdots$ n odd

$h_x = \underbrace{h_0 + x^2 h_2 + \cdots + x^{n-2} h_{n-2}}_{\text{even terms}} + x^{n} \log x \, h_{n,1} + x^n h_n + \cdots$ n even

- A priori, this is a formal expansion

[CDLS] [BH] showed the formal expansion is attained by CCE metrics.

(using an inductive arguement + elliptic regularity of D.O., one can show CCE metrics are polyhomogeneous : admit an expansion involving $(x)^{i}$ and $(\ln x)^{j}$.)

2.4. Obstruction to a smooth expansion.

- When n even, there's an obstruction arising from the log term. i.e. $h_{n,1}$ may not vanishing

One can compute leading order term of the obstruction is

[GH] $$\Delta^{(n/2)-2} \left(P_{ij,k}{}^{k}{}_{,} - P_k{}^{k}{}_{,ij} \right)$$

where P is the Schouten tensor of h.

2.5. Dirichlet-to-Neuman map

- Here we restricte to n odd, where h_x has a smooth expansion

- The conformal infinity gives Dirichlet data on $[0,\varepsilon)_x \times M$. $\hookrightarrow h_0 = h_x|_{x=0}$

  h_0 determines $h_2, h_4, \cdots h_{n-1}$.
  (This is because we can inductively solve the equation $Ric_g + ng = O(x^k)$ up to $k = n-1$).

- However, the coefficient h_n is not locally determined by h_0}. $\hookrightarrow$ Neumann data

- (The Dirichlet-to-Neumann map N we consider here is $N : \phi \mapsto \psi$ later)
  We call a pair (ϕ, ψ) a D-to-N relation, which is a collection of pairs in $S^2(TX)$.
  We denote $(\phi, \psi) \in N$.

- N is equivariant under diffeomorphism
  $$(\Phi^* \phi, \Phi^* \psi) \in N$$
  N is equivariant under conf. transformation
  $$(e^{2u}\phi, e^{2(n-1)u}\psi) \in N.$$

- The D-to-N relation gives a D-to-N map if there exists a unique CCE metric

$$g = \frac{dx^2 + h_x}{x^2}$$

(up to a diffeo Φ s.t. $\Phi|_M = id$) s.t.

$$h_0 = \phi, \quad h_n = \psi.$$

- In general, $\exists$ and $!$ are not guarenteed. So N is not well-defined.

- However, we have seen that when h is $C^{2,\alpha}$-closed to g_{S^n}, N is well-defined.
In such case, we can compute the principal symbol of dN.

$$\sigma_n(dN) = \sigma_n\left(2^{-n} \frac{\Gamma(-\frac{n}{2})}{\Gamma(\frac{n}{2})} \Delta^{n/2}\right)$$

- In general, if we assume N is well-defined,

[Wan] $$\sigma_n(dN) = \sigma_n\left(2^{-n} \frac{\Gamma(-\frac{n}{2})}{\Gamma(\frac{n}{2})} \Delta_{h_0}^{n/2} \circ \left(tf - tf(\delta^*(\delta tf\, \delta^*)^+)\, \delta\, tf\right)\right)$$

<u>idea</u>: 1. compute σ_n for h_0 that is $\begin{cases}\text{trace free} \\ \text{divergence free}\end{cases}$

2. for a general h_0, modified h_0 by

$$\tilde{h}_0 = h_0' + u h_0 + \delta_{h_0}^* w, \quad u = \frac{1}{n}(tf\, h_0' - tf\, \delta^* w).$$

Ref.

[FG] Fefferman + Graham, 1985
"Conformal invariants".

[GH] Graham, Hirach, 2005
"The ambient obstruction tensor and Q-curvature".

[GL] Graham, Lee, 1991
"Einstein metrics with prescribed conformal infinity on the ball".

[BH] Biquard, Herzlich, 2014
"Analyse sur un demi-espace hyperbolique et poly-homogénéité locale".

[CDLS] Chruściel, Delay, Lee, Skinner, 2005
"Boundary regularity of conformally compact Einstein metrics".

[Wan] Wang, 2009
"Dirichlet-to-Neumann map for Poincaré-Einstein in Even Dimensions".